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# NUMERICAL SOLUTION OF RADIAL HEAT CONDUCTION IN AN INFINITELY LONG CYLINDER

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## ABSTRACT
Use of modern computational techniques supporting solution of imposed boundary condition in heat transfer problems is increasing day by day. However determining the appropriate technique in the context of problem statement, complexity of geometry, simplicity of usage and exacting solution are the challenge researchers are facing. In this paper, an attempt has been made to find solution through different computational methods for a one dimensional heat flow problem in steady state. Finite Difference Method (FDM), Finite volume method (FVM) and Finite Element method (FEM) have been used and a comparative analysis has been considered to arrive at a desired exactness of the solution. The calculated values from each of the methods are compared and analyzed. A MATLAB implementation of the numerical solution has also been provided. Numerical results are presented for a sequence of finer meshes, and the dependency of the truncation error on mesh size is verified.

**Keywords:** TDMA, FDM, FVM, FEM

## 1. INTRODUCTION

Solutions to conduction problems to establish the temperature distribution can be derived from the appropriate differential equations with imposed boundary conditions. Analysis of such solutions is better done through various computational techniques. Antar [4] offered a simplified numerical solution of unsteady heat conduction problem in a short cylinder. This paper proposes simpler solution for heat conduction problem in one –dimensional form and compares the results obtained in each method (FDM, FVM, FEM) [5], [12] for different meshes.

The finite difference method (FDM) [7] is based on the differential equation of the heat conduction, which is transformed into a difference equation. The temperature values are calculated at the nodes of the network. Using this method, convergence and stability problem can appear. Han et al. [5] used FDM method with FEM for analysis of one-dimensional fin.

Dhawan et al. [2] studied the heat conduction problem in an aluminium plate using finite element method and got very good results compared to the exact solution. The finite element method (FEM) [7] is based on the integral equation of the heat conduction. This is obtained from the differential equation using variational calculus. In first case the temperature values are calculated on the finite elements. Then, based on these partial solutions, the solution for the entire volume is determined. Using this method the whole surface has been divided into elements and fields with unregulated border. In this paper the temperature distribution is analyzed in the radial direction of a solid cylinder. The practical application is in the electrical wires where heat is generated in the wire and the same heat it needed to be dissipated to the surrounding so that the melting of the wire is avoided.

The cylinder in this case is infinitely long with isotropic physical characteristics and it has been assumed that the heat is being transferred in the radial direction. Heat conduction in the axial direction is neglected.

## 2. THEORY

### 2.1 Governing Differential Equation

The general heat conduction problem in a 3 dimensional plane with generation and with time variation is given by the equation given below.

Governing Differential Equation:

$$k\left\{\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right\}+g=\rho C\frac{\partial T}{\partial \tau} \tag{1}$$

The similar governing differential Equation for cylinder is:

$$k\left\{\frac{1}{r}\frac{d}{dr}r\frac{dT}{dr}\right\}+g=\rho C\frac{dT}{d\tau} \tag{2}$$

Neglecting d the heat conduction in the axial direction the equation becomes one dimension, assuming axi-symmetry.

. The similar governing differential equation (GDE) for sphere is given as:

$$k\left\{\frac{1}{r^2}\frac{d}{dr}r^2\frac{dT}{dr}\right\}+g=\rho C\frac{dT}{dt} \quad (3)$$

In this paper, focus has been given on the GDE for cylinder.

### 2.2 Boundary Conditions

**a. Symmetric Boundary Condition**

This treatment of the boundary condition corresponds to the physical assumption that, on the two sides of the boundary, the same physical processes exist. The variable values at the same distance from the boundary at the two sides are the same. The function of such a boundary is that of a mirror that can reflect all the fluctuations generated in the simulation region.

In this case, under symmetry boundary condition at the center i.e. at r=0 , $\frac{dT}{dr}=0$

**b. Robbin's Boundary Condition**

This treatment of the boundary is done when the boundary is exposed or open to the atmosphere. Hence in this kind of boundary condition the conductive heat flux and convective heat flux are made equal. This kind of boundary condition is also called as mixed type boundary condition.

Considering the above boundary condition for the cylinder,

At r = R,

$$-k\frac{dT}{dr}=h(T_s-T_\infty) \quad (4)$$

### 2.3 Exact Solution

Integrating the governing differential equation within the boundary conditions we obtain:

$$\int\frac{d}{dr}r\left(\frac{dT}{dr}\right)dr+\int\frac{g}{k}r\,dr=C_1 \quad (5)$$

Putting boundary condition 1: at r=0, $\left(\frac{dT}{dr}\right)=0$, we get $C_1$=0.

Again integrating: $T+\frac{g}{4k}r^2=C_2$

Using boundary condition 2, at r=R $-k\frac{dT}{dr}=h(T_s-T_\infty)$

We get $C_2=T_\infty+\frac{g}{4k}R^2+\frac{gR}{2h}$

So the exact temperature profile is:

$$T=T_\infty+\frac{gR}{2h}+\frac{gR^2}{4k}\left(1-\frac{r^2}{R^2}\right) \quad (6)$$

## 3. NUMERICAL SOLUTION

### 3.1 Finite Difference Method (FDM)

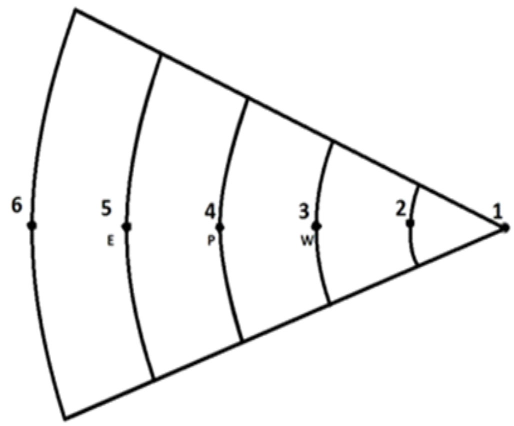


Fig 1. Geometry of cylinder showing 6 different nodes for the finite difference method

As shown in Fig 1. , the symmetrical cylinder solid structure is divided into six different nodes for the finite difference method. The general governing differential equation is discretised using FDM is as follows:

For n equal divisions, there will be m nodes where m=n+1

For node i=1:

The governing differential equation can be written as:

$$\frac{d^2T}{dr^2}+\frac{1}{r}\frac{dT}{dr}+\frac{g}{k}=0 \;(0<r\leq R) \quad (7)$$

To avoid singularity at the origin $\left(\frac{1}{r}\frac{dT}{dr}=\frac{0}{0}\right)$, L-Hospitals rule is used to get the discretized equation at i=1 as:

$$4T_i=4T_{i+1}+\frac{g\Delta r^2}{k}$$

For i = 2 to m-1:

$$2T_i=\left(1+\frac{1}{2(i-1)}\right)T_{i+1}+\left(1-\frac{1}{2(i-1)}\right)T_{i-1}+\frac{g\Delta r^2}{k}$$

For i= m:

$$\left[2+\left(1+\frac{1}{2(m-1)}\right)\left(\frac{2\Delta rh}{k}\right)\right]T_m=2T_{m-1}+\left(1+\frac{1}{2(m-1)}\right)\left(\frac{2\Delta rh}{k}\right)T_\infty+\frac{g\Delta r^2}{k}$$

### 3.2 Finite Volume Method (FVM)

Finite Volume method is another method for numerical calculation of the differential equation. In this method the entire volume is divided into smaller volumes and inlet flux is equated with the outlet flux giving an approximate numerical approach to the temperature at each mesh [8].

The FVM equations for the solid cylinder considered in this paper are:

$$\int_w^e\frac{1}{r}\frac{d}{dr}r\left(\frac{dT}{dr}\right)dV+\int_w^e\frac{g}{k}=0$$

$$\left[r\frac{dT}{dr}\right]_w^e=\frac{g}{k}\left[\frac{r^2}{2}\right]_w^e \quad (8)$$

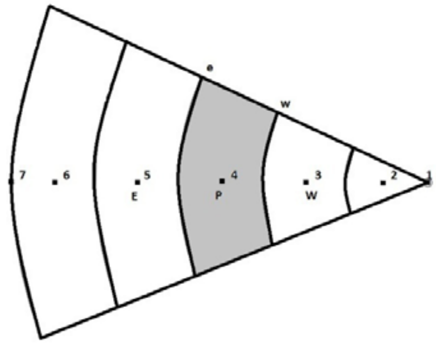


Fig 2. Geometry of cylinder showing 7 different shell centers for the finite volume method

For n equal divisions, there will be m shell centers where m=n+2

For i=1: $T_i = T_{i+1}$

For i=2: $T_i = T_{i+1} + \frac{g\Delta r^2}{2k}$

For i= 3 to m-2:

$$(2i-3)T_i = (i-1)T_{i+1} + (i-2)T_{i-1} + (2i-3)\frac{g\Delta r^2}{2k}$$

For i=m-1:

$$(3i-4)T_i = 2(i-1)T_{i+1} + (i-1)T_{i-1} + (2i-3)\frac{g\Delta r^2}{2k}$$

For i=m:

$$\left(2+\frac{h\Delta r}{k}\right)T_m = 2T_{m-1} + \frac{h\Delta r}{k}T_\infty$$

### 3.3 Solution Technique

The governing differential equation has been discretized using both FDM and FVM. Both the equations are in the form: $a_iT_i=b_iT_{i+1}+c_iT_{i-1}+d_i$

So, Tri-Diagonal Matrix Algorithm (TDMA) [14] is appropriate to solve the discretized equations.

The corresponding MATLAB Code is appended. The TDMA solution follows forward elimination with back-substitution using the equation: $T_i=P_iT_{i+1}+Q_i$, where

$$P_i = \frac{b_i}{a_i - c_iP_{i-1}} \text{ and } Q_i = \frac{d_i + c_iQ_{i-1}}{a_i - c_iP_{i-1}}$$

### 3.4 Finite Element Method (FEM)

Finite Element Method is another method for numerical solution the differential equation. ANSYS 13.0 well established commercial software is used in this paper for required analysis and the details are produced in the results and discussion section.

## 4. RESULTS AND DISCUSSION

The one-dimensional heat conduction problem considered in this paper has been solved using exact solution and then compared with that obtained through FDM, FVM and FEM.

Following parameters have been considered for solutions in all techniques as cited above:

Table 1: Parameter matrix

| ***Parameter*** | ***Value*** |
|---|---|
| Radius (m) | 0.05 |
| Generation (Watt/m$^3$) | 4x10$^6$ |
| k (Watt/mK) | 40 |
| h (Watt/m$^2$K) | 400 |
| $T_\infty$ ($^o$C) | 20 |

Table 2: Comparative results of temperature distribution obtained through FDM, FVM and exact solution

| **Radius (m)** | **FDM ($^0$C)** | **FVM ($^0$C)** | **Exact ($^0$C)** |
|---|---|---|---|
| 0.000 | 332.500 | 332.500 | 332.500 |
| 0.005 | 331.875 | 332.500 | 331.875 |
| 0.010 | 330.000 | 330.000 | 330.000 |
| 0.015 | 326.875 | 327.500 | 326.875 |
| 0.020 | 322.500 | 322.500 | 322.500 |
| 0.025 | 316.875 | 317.500 | 316.875 |
| 0.030 | 310.000 | 310.000 | 310.000 |
| 0.035 | 301.875 | 302.500 | 301.875 |
| 0.040 | 292.500 | 297.500 | 292.500 |
| 0.045 | 281.875 | 292.500 | 281.875 |
| 0.050 | 270.000 | 270.000 | 270.000 |

As can be seen from the above table (Table 2), the FDM solution and the exact solution are exactly match without any error. This is because of the use of second order accurate finite difference for first order derivative in which case the truncation error is found to be zero [7].

The difference in results obtained through exact solution and that through FVM is due to the use of first order accurate difference equation, for first order derivative.

The results represented in Table 2 have also been shown in the form of graphical representation.

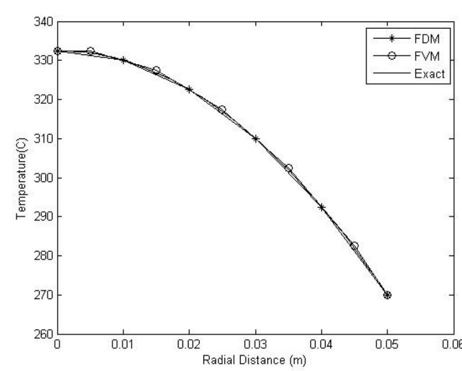


Fig 3. Comparison of FDM, FVM and exact solution having six nodes

One of the features of FDM and FVM is that the relative error between the analytic method and these two methods decreases as the numbers of nodes (shell centers) increase. This has been established in the following graph in which the results have been plotted by solving the heat conduction problem in FDM and FVM with 100 nodes.

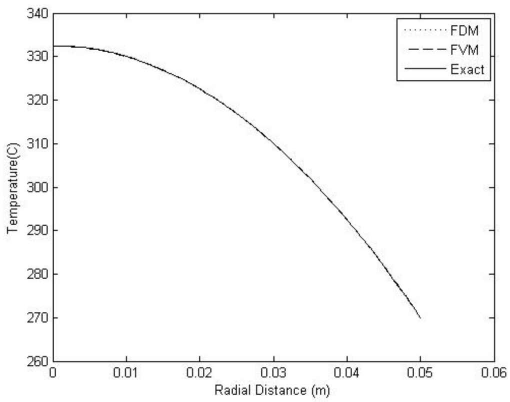


Fig 4. Comparison of FDM, FVM and exact solution with 100 nodes

**Results obtained through FE Analysis**

ANSYS has been used for analysis of temperature distribution through FEM.

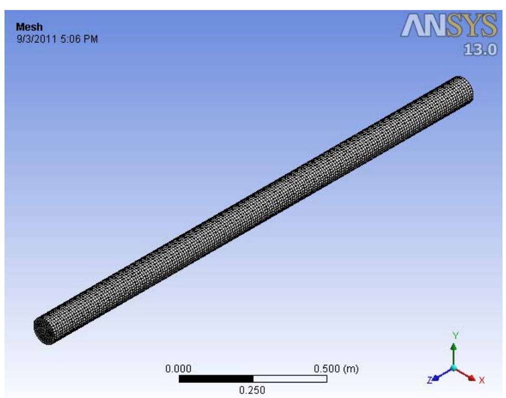


Fig 5. Mesh on the entire rod

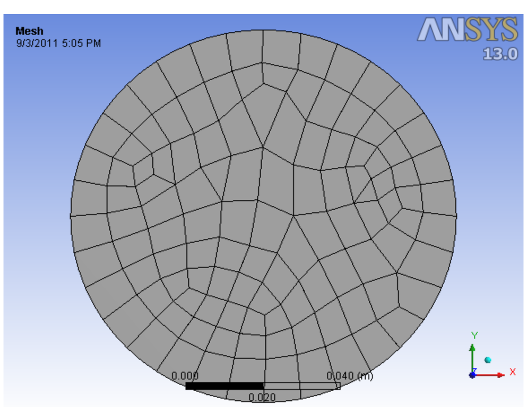


Fig 6. Mesh on the cross-section

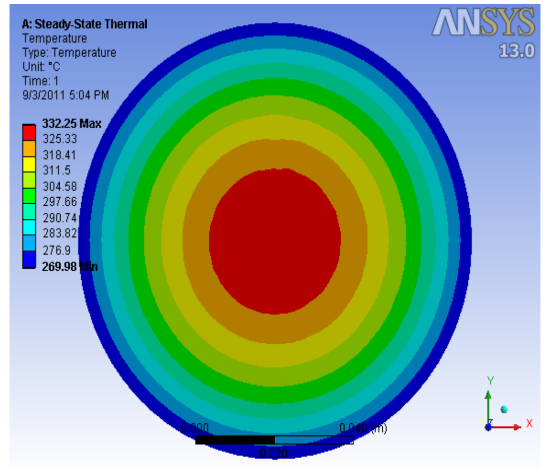


Fig 7. Cross –sectional temperature contour

Above figure (Fig. 7) shows no direct matching the results obtained through exact solutions due to the non-uniformity in meshing.

## 5. CONCLUSION

The numerical modelling with FEM, FVM and FDM represents an efficient way to obtain temperature distribution in steady state conductive heat transfer processes. They also render the solution better to be automated through computer.

The numerical computation of the temperature field, on the basis of FDM offer better results for regular geometry, whereas FEM and FVM can offer better results for irregular geometry and complex boundary conditions. Also, choice of finer grids which requires high computing capability can remove approximation errors to larger extent.

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## 7. NOMENCLATURE

| ***Symbol*** | ***Parameter*** | ***Units*** |
|---|---|---|
| r | Radius | Meter |
| T | Temperature | $^oC$ |
| $\rho$ | Density | $kg/m^3$ |
| t | Time | sec |
| C | Specific Heat Capacity | kJ/Kg-K |
| k | Thermal Conductivity | Watt/m-K |
| h | Convective heat Transfer Co-efficient | $Watt/m^2$-K |
| g | Generation term | $Watt/m^3$ |
| $T_s$ | Surface temperature | $^oC$ |
| $T_\infty$ | Surrounding temperature | $^oC$ |
| i | Node number | No unit |

## APPENDIX

**MATLAB CODE:**

```
clc
clear all
n=input('Number of nodes:');
r=input('Radius of the Cylinder(m):');
k=input('Thermal Conductivity(Watt/mK):');
h=input('Convective heat transfer
Co-efficient(Watt/m2):');
g=input('Generation term(Watt/m3):');
Ti=input('Ambient air temperature(0C):');
m=n+1;
dx=r/n;
clc
disp(' FINITE DIFFERENCE SOLUTION');
a(:,:)=0;
for i=2:m-1


 a(i,1)=2;
   a(i,2)=(1+(1/(2*(i-1))));
   a(i,3)=(1-(1/(2*(i-1))));
   a(i,4)=g*dx*dx/k;
end
a(1,1)=4;
a(1,2)=4;
a(1,4)=g*dx*dx/k;
a(m,1)=((2)+(1+(1/(2*(m-1))))*(2*dx*h/k));
a(m,3)=2;
a(m,4)=((1+(1/(2*(m-1))))*(2*dx*h/k)*Ti)+(g*dx*dx/k
);
for i=1:m
  if(a(i,3)==0)
      p(i)=a(i,2)/a(i,1);
   end
   if(a(i,3)~=0)
      p(i)=a(i,2)/(a(i,1)-(a(i,3)*p(i-1)));
   end
end
%disp('Value of P');
```

```
%disp(p);
%calculation of q
for i=1:m
   if(a(i,3)==0)
      q(i)=a(i,4)/a(i,1);
   end
   if(a(i,3)~=0)
   q(i)=(a(i,4)+(a(i,3)*q(i-1)))/(a(i,1)-(a(i,3)*p(i-1)));
   end
end
%disp('Value of Q'),q
%calculating u
T(1:5)=0;
for i=m:-1:1
   if(i==m)
      T(i)=q(i);
   end
   if(i~=m)
      T(i)=(p(i)*T(i+1))+ q(i);
   end
end
T
for i=1:m
x(i)=(i-1)*dx;
end
plot(x,T(1:m));
hold on
disp(' FINITE VOLUME SOLUTION');
m=n+2;
a(:,:)=0;
for i=3:m-2
   a(i,1)=(2*i)-3;
   a(i,2)=i-1;
   a(i,3)=i-2;
   a(i,4)=g*((2*i)-3)*dx*dx/(2*k);
end
a(1,1)=1;
a(1,2)=1;
a(2,1)=1;
a(2,2)=1;
a(2,4)=g*dx*dx/(2*k);
a(m-1,1)=3*(m-1)-4;
a(m-1,2)=2*(m-2);
a(m-1,3)=m-3;
a(m-1,4)=g*((2*(m-1))-3)*dx*dx/(2*k);
a(m,1)=2+(h*dx/k);
a(m,3)=2;
a(m,4)=h*dx*Ti/k;
for i=1:m
  if(a(i,3)==0)
      p(i)=a(i,2)/a(i,1);
   end
   if(a(i,3)~=0)
      p(i)=a(i,2)/(a(i,1)-(a(i,3)*p(i-1)));
   end
end
% disp('Value of P');
% disp(p);
%calculation of q
for i=1:m
   if(a(i,3)==0)
      q(i)=a(i,4)/a(i,1);
   end
   if(a(i,3)~=0)
   q(i)=(a(i,4)+(a(i,3)*q(i-1)))/(a(i,1)-(a(i,3)*p(i-1)));
   end
end
% disp('Value of Q'),q
%calculating u
T(1:7)=0;
for i=m:-1:1
   if(i==m)
      T(i)=q(i);
   end
   if(i~=m)
      T(i)=(p(i)*T(i+1))+ q(i);
   end
end
T
x(1)=0;
x(2)=dx/2;
x(m)=r;
for i=3:m-1
x(i)=x(i-1)+dx;
end
plot(x,T(1:m));
hold on
(g*r/(2*h))+(g*r*r/(4*k))+Ti
a=[-g/(4*k) 0 (g*r/(2*h))+(g*r*r/(4*k))+Ti];
i=1;
k1(1)=0;
for x=0:0.001:r-0.001
   i=i+1;
   k1(i)=k1(i-1)+0.001;
end
plot(k1,polyval(a,k1));
```